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THE COORDINATE-TRANSFORMATION EQUATIONS FOR A PILOTED FLIGHT SIMULATOR WITH SEVERAL DEGREES OF FREEDOM

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SUMMARY

A method for developing coordinate-transformation equations for a multiple-degree-of-freedom flight simulator is presented. The equations as developed are applicable in particular to the NASA five-degree-of-freedom piloted flight simulator; in general, however, the method of their development is applicable to transformation equations for other, similar simulators of fewer or of more degrees of freedom.

Because the NASA simulator has a very limited range of displacement in one of its modes of motion the equations are written for four degrees of freedom instead of for five. Examination of the singularities of the equations showed it possible to reproduce any combination of four of the six components of motion, three angular and three linear, of the vehicle being simulated. It was found that, in most cases, there is more than one way to simulate each combination, the most desirable way determined by the restrictions imposed by the singularities of the equations.

INTRODUCTION

Piloted flight simulators are used to investigate problems associated with the control of humanly piloted aircraft and spacecraft, both existing and proposed. These simulators impose upon the pilot a partial reproduction of stimuli pertinent to his control of the vehicle. In the main, these are stimuli of the visual and of the kinesthetic senses. Existing simulators range from the relatively simple, which simulate only the visual stimulus from the panel instruments, to the highly complex, which simulate both the kinesthetic stimulus from the motion of the vehicle and the visual stimulus from the scene outside the cockpit.

The Ames Research Center has recently constructed a five-degree-of-freedom motion simulator, a piloted centrifuge, capable of reproducing components of both angular and linear motion. A study was made to examine how faithfully linear acceleration and angular velocity can be reproduced on the Ames simulator, and to determine how to program a desired simulation. This report was drawn from that study.

The present report describes the method used to develop and analyze the kinematics equations of the simulator. Although the details of the geometry and of the partitioning of the equations apply explicitly to the Ames simulator, the method is general and can be used to study the kinematic simulations allowed on any piloted centrifuge with several degrees of freedom.

The analysis begins with a description of the physical configuration of the Ames simulator. The desired relationships between simulator kinematics and linear acceleration and angular velocity are next determined. The resulting expressions are rearranged so that the limitations imposed on the simulation by the mathematical constraints of the transformation equations can be explored. A procedure is then described for A
determining initial conditions to be used in programming a simulation 4
on an analog computer. Finally, the relationships between the axes of 9
the simulator cockpit and the axes of the simulated vehicle are examined. 7

NOTATION

$\bar{A}$ total vector linear acceleration of the center of rotation of the simulator cockpit

a_V magnitude of the component of $\bar{A}$ in the direction $\bar{3}^C$

a_{iG} magnitude of the component of $\bar{A}$ in the direction $\bar{i}^G$, $i = 1, 2, 3$

C axes set of three right-hand orthogonal axes fixed in the centrifuge arm; origin at the center of rotation of the cockpit; orientation defined by $\bar{1}^C$, $\bar{2}^C$, $\bar{3}^C$

G axes set of three right-hand orthogonal axes fixed in the simulator cockpit; origin at the center of rotation of the cockpit; orientation defined by $\bar{1}^G$, $\bar{2}^G$, $\bar{3}^G$

J axes set of three right-hand orthogonal axes fixed in the inner gimbal; origin at the center of rotation of the simulator cockpit; orientation defined by $\bar{1}^J$, $\bar{2}^J$, $\bar{3}^J$

L axes set of three right-hand orthogonal axes fixed in the earth; origin at the intersection of the centrifuge arm with the axis of rotation of the centrifuge arm; orientation defined by $\bar{1}^L$, $\bar{2}^L$, $\bar{3}^L$

M axes set of three right-hand orthogonal axes fixed in the outer gimbal; origin at the center of rotation of the cockpit; orientation defined by $\bar{1}^M$, $\bar{2}^M$, $\bar{3}^M$

$\bar{1}^C$	vector of unit magnitude, forming a right-hand orthogonal triad with $\bar{2}^C$ and $\bar{3}^C$
$\bar{2}^C$	vector of unit magnitude, orthogonal to $\bar{1}^C$ and $\bar{3}^C$, along the centrifuge arm, positive from the C axis; origin toward the axis of rotation of the arm
$\bar{3}^C$	vector of unit magnitude, parallel to $\bar{3}^L$, along the outer gimbal drive axis; positive downward
$\bar{1}^G$	identical to $\bar{1}^J$
$\bar{2}^G$	vector of unit magnitude fixed in the cockpit, forming a right-hand orthogonal triad with $\bar{1}^G$ and $\bar{3}^G$
$\bar{3}^G$	vector of unit magnitude, fixed in the cockpit, forming a right-hand orthogonal triad with $\bar{1}^G$ and $\bar{2}^G$
$\bar{1}^J$	vector of unit magnitude along the cockpit drive axis, forming a right-hand orthogonal triad with $\bar{2}^J$ and $\bar{3}^J$
$\bar{2}^J$	identical to $\bar{2}^M$
$\bar{3}^J$	vector of unit magnitude forming a right-hand orthogonal triad with $\bar{2}^J$ and $\bar{1}^J$
$\bar{3}^L$	vector of unit magnitude along the axis of rotation of the centrifuge arm; positive downward
$\bar{1}^M$	vector of unit magnitude forming a right-hand orthogonal triad with $\bar{2}^M$ and $\bar{3}^M$
$\bar{2}^M$	vector of unit magnitude along the inner gimbal drive axis, forming a right-hand orthogonal triad with $\bar{1}^M$ and $\bar{3}^M$
$\bar{3}^M$	identical to $\bar{3}^C$
γ_1	angular displacement between $\bar{2}^J$ and $\bar{2}^G$ and between $\bar{3}^J$ and $\bar{3}^G$, in the plane perpendicular to $\bar{1}^J$ and positive according to the right-hand rule with respect to $\bar{1}^J$; zero when $\bar{2}^J = \bar{2}^G$, $\bar{3}^J = \bar{3}^G$
γ_2	angular displacement between $\bar{1}^M$ and $\bar{1}^J$ and between $\bar{3}^M$ and $\bar{3}^J$ in the plane perpendicular to $\bar{2}^M$ and positive according to the right-hand rule with respect to $\bar{2}^M$; zero when $\bar{1}^M = \bar{1}^J$, $\bar{3}^M = \bar{3}^J$

A
4
9
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γ_3 angular displacement between $\overline{1}^C$ and $\overline{1}^M$ and between $\overline{2}^C$ and $\overline{2}^M$ in the plane perpendicular to $\overline{3}^C$ and positive according to the right-hand rule with respect to $\overline{3}^C$; zero when $\overline{1}^C = \overline{1}^M$, $\overline{2}^C = \overline{2}^M$

$\dot{\gamma}_i$ time rate of change of γ_i, i = 1, 2, 3

$\dot{\lambda}$ angular velocity of the centrifuge arm about $\overline{3}^L$, positive according to the right-hand rule with respect to $\overline{3}^L$

ρ length of centrifuge arm, perpendicular distance between $\overline{3}^L$ and $\overline{3}^C$

$\overline{\Omega}$ total vector angular velocity about the center of rotation of the simulator cockpit

ω_{iG} magnitude of the component of $\overline{\Omega}$ about $\overline{i}^G$, positive according to the right-hand rule with respect to $\overline{i}^G$, i = 1, 2, 3

A
4
9
7

THE SIMULATOR

Figure 1 is a photograph of the Ames five-degree-of-freedom piloted flight simulator. The cockpit is set in gimbals so that it can rotate independently about three axes; the cockpit and gimbal assembly is mounted on the end of the centrifuge arm, which can rotate in a horizontal plane and thereby impose centrifugal acceleration upon the cockpit. In addition, the cockpit (and gimbal assembly) can accelerate vertically with reference to the arm. However, because of the very limited range of vertical displacement, vertical acceleration of significant magnitude can be imposed only in pulses of very short duration. The simulator can, of course, additionally provide the visual stimulus from the cockpit panel instruments, but here we will be concerned with only the kinesthetic stimulus. The transformation equations relate the coordinates of the angular and the linear accelerations along and about cockpit axes to the coordinates along and about the axes of freedom of the simulator. Because control over the vertical acceleration is quite restricted, it is anticipated that techniques for using this vertical degree of freedom, to null spurious transient accelerations, will be developed empirically in practice; therefore, in the equations the simulator is treated as having only four degrees of freedom instead of five. The vertical acceleration is treated as an independent fixed input to the system.

THE COORDINATE-TRANSFORMATION EQUATIONS

The coordinate systems assigned to the several independent parts of the simulator are indicated in figure 2, a schematic diagram of the simulator. The notation and the coordinate-transformation procedures of reference 1 will be followed.

The degrees of freedom of the simulator are as follows: The centrifuge arm can rotate about $\bar{3}^L$ with angular velocity of magnitude $\dot{\lambda}$. The outer gimbal can rotate about $\bar{3}^C$ with angular velocity of magnitude $\dot{\gamma}_3$, the inner gimbal about $\bar{2}^M$ with magnitude $\dot{\gamma}_2$, and the cockpit about $\bar{1}^J$ with magnitude $\dot{\gamma}_1$. Limited, linear travel of the gimbal assembly (outer gimbal, inner gimbal, and cockpit) is possible along the unit vector $\bar{3}^C$. However, as mentioned before, in the development of the equations this mode of motion is not treated as a degree of freedom. The acceleration along $\bar{3}^C$ is of magnitude a_v.

In the discussions which follow, the axes $\bar{3}^L$, $\bar{3}^C$, $\bar{2}^M$, $\bar{1}^J$ will comprise the drive-axes system of the simulator.

The total vector angular velocity of the cockpit, expressed in coordinates along drive axes, is

$$\bar{\Omega} = \dot{\gamma}_1\bar{1}^J + \dot{\gamma}_2\bar{2}^M + \dot{\gamma}_3\bar{3}^C + \dot{\lambda}\bar{3}^L$$

and the total vector linear acceleration is

$$\bar{A} = \rho\ddot{\lambda}\bar{1}^C + \rho(\dot{\lambda})^2\bar{2}^C + a_v\bar{3}^C$$

The vectors $\bar{\Omega}$ and $\bar{A}$ can also be expressed as sums of components along the three cockpit axes, the $\underline{G}$ axes. Let ω_{iG} be the coordinate of the component of $\bar{\Omega}$ about the $\bar{i}^G$ axis, and a_{iG} be the coordinate of the component of $\bar{A}$ along $\bar{i}^G$, $i = 1, 2, 3$. Then

$$\bar{\Omega} = \omega_{1G}\bar{1}^G + \omega_{2G}\bar{2}^G + \omega_{3G}\bar{3}^G$$

$$\bar{A} = a_{1G}\bar{1}^G + a_{2G}\bar{2}^G + a_{3G}\bar{3}^G$$

The transformation equations which have to be developed must relate the coordinates along simulator-drive axes to the coordinates along cockpit axes. However, because there are only four degrees of freedom in the system, there can be a total of, at most, four mutually independent coordinates which describe both $\bar{A}$ and $\bar{\Omega}$; that is, among the six G axes coordinates, ω_{1G}, ω_{2G}, ω_{3G}, a_{1G}, a_{2G}, a_{3G}, of $\bar{A}$ and $\bar{\Omega}$, four, at most, are mutually independent, which four being a matter of choice. The remaining two coordinates are automatically determined once the four independent coordinates have been chosen.

Generally the simulation program will be set up as follows. The simulator cockpit axes (G axes) will be set to correspond in a particular way with the axes of the vehicle to be simulated (according to a plan which will be developed along with the equations). The four G axes coordinates of $\bar{A}$ and $\bar{\Omega}$ to be mutually independent will be chosen to correspond to the four coordinates of the angular velocity and the linear acceleration, of the actual vehicle, which are most pertinent to the problem being studied and, hence, are to be faithfully reproduced. Then the transformation equations will be solved for the simulator-drive coordinates $\dot{\gamma}_1$, $\dot{\gamma}_2$, $\dot{\gamma}_3$, and $\rho\ddot{\lambda}$ as functions of the four independent G axes coordinates and, in addition, for the two G axes coordinates of $\bar{A}$ and $\bar{\Omega}$ which were not chosen to be independent. These latter two define the spurious stimulus to which the simulator pilot is subjected, that is, that part of the total simulated stimulus which is not a representation of any stimulus of the actual flight.

In the discussions that follow, the four G axes coordinates which are to be mutually independent will be called simply the independent variables, and the four simulator-drive-axes coordinates together with the two dependent G axes coordinates will be called the dependent variables. The dependent variables will always include $\dot{\gamma}_1$, $\dot{\gamma}_2$, $\dot{\gamma}_3$, and $\rho\dot{\lambda}$ (or, as it will later turn out, $\rho\ddot{\lambda}$).

EQUATIONS FOR REPRODUCING THE TOTAL ANGULAR VELOCITY AND ONE COMPONENT OF LINEAR ACCELERATION

Development of the Equations

We may derive, by the method of reference 1, pages 6 and 7, the general form of the transformation matrix that relates the coordinates of $\bar{\Omega}$ in drive axes to the coordinates in G axes:

$$\begin{bmatrix} \omega_{1G} \\ \omega_{2G} \\ \omega_{3G} \end{bmatrix} = \begin{bmatrix} \bar{1}^J \cdot \bar{1}^G & \bar{2}^M \cdot \bar{1}^G & \bar{3}^C \cdot \bar{1}^G & \bar{3}^L \cdot \bar{1}^G \\ \bar{1}^J \cdot \bar{2}^G & \bar{2}^M \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G & \bar{3}^L \cdot \bar{2}^G \\ \bar{1}^J \cdot \bar{3}^G & \bar{2}^M \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G & \bar{3}^L \cdot \bar{3}^G \end{bmatrix} \begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \dot{\lambda} \end{bmatrix} \qquad (1)$$

Similarly, for the coordinates of linear acceleration:

$$\begin{bmatrix} a_{1G} \\ a_{2G} \\ a_{3G} \end{bmatrix} = \begin{bmatrix} \bar{1}^C \cdot \bar{1}^G & \bar{2}^C \cdot \bar{1}^G & \bar{3}^C \cdot \bar{1}^G \\ \bar{1}^C \cdot \bar{2}^G & \bar{2}^C \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G \\ \bar{1}^C \cdot \bar{3}^G & \bar{2}^C \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G \end{bmatrix} \begin{bmatrix} \rho\ddot{\lambda} \\ \rho(\dot{\lambda})^2 \\ a_v \end{bmatrix} \tag{2}$$

It will simplify somewhat the matrices of equations (1) and (2) if we write as unity or as zero those dot products which are unity or zero. The dot product of two unit vectors is the cosine of the angle between them. Therefore, $\bar{1}^J \cdot \bar{1}^G = 1$ and $\bar{1}^J \cdot \bar{2}^G = \bar{1}^J \cdot \bar{3}^G = \bar{2}^M \cdot \bar{1}^G = 0$ (refer to fig. 2). Then equation (1) becomes

$$\begin{bmatrix} \omega_{1G} \\ \omega_{2G} \\ \omega_{3G} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \bar{3}^C \cdot \bar{1}^G & \bar{3}^L \cdot \bar{1}^G \\ 0 & \bar{2}^M \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G & \bar{3}^L \cdot \bar{2}^G \\ 0 & \bar{2}^M \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G & \bar{3}^L \cdot \bar{3}^G \end{bmatrix} \begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \dot{\lambda} \end{bmatrix} \tag{3}$$

We can rewrite equation (2)

$$\begin{bmatrix} \bar{1}^C \cdot \bar{1}^G & -1 & 0 & 0 \\ \bar{1}^C \cdot \bar{2}^G & 0 & -1 & 0 \\ \bar{1}^C \cdot \bar{3}^G & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \rho\ddot{\lambda} \\ a_{1G} \\ a_{2G} \\ a_{3G} \end{bmatrix} = \begin{bmatrix} -\bar{2}^C \cdot \bar{1}^G & -\bar{3}^C \cdot \bar{1}^G \\ -\bar{2}^C \cdot \bar{2}^G & -\bar{3}^C \cdot \bar{2}^G \\ -\bar{2}^C \cdot \bar{3}^G & -\bar{3}^C \cdot \bar{3}^G \end{bmatrix} \begin{bmatrix} \rho(\dot{\lambda})^2 \\ a_v \end{bmatrix} \tag{2a}$$

We can rewrite (3)

$$\begin{bmatrix} 1 & 0 & \bar{3}^C \cdot \bar{1}^G & -1 & 0 & 0 \\ 0 & \bar{2}^M \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G & 0 & -1 & 0 \\ 0 & \bar{2}^M \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \omega_{1G} \\ \omega_{2G} \\ \omega_{3G} \end{bmatrix} = \begin{bmatrix} -\bar{3}^L \cdot \bar{1}^G \\ -\bar{3}^L \cdot \bar{2}^G \\ -\bar{3}^L \cdot \bar{3}^G \end{bmatrix} [\dot{\lambda}] \tag{3a}$$

We can then combine equations (2a) and (3a) into one set of six equations:

$$\begin{bmatrix} 1 & 0 & \bar{3}^C \cdot \bar{1}^G & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \bar{2}^M \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & \bar{2}^M \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{1}^G & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{2}^G & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{3}^G & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \rho\ddot{\lambda} \\ a_{1G} \\ a_{2G} \\ a_{3G} \\ \omega_{1G} \\ \omega_{2G} \\ \omega_{3G} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\bar{3}^L \cdot \bar{1}^G & 0 \\ 0 & -\bar{3}^L \cdot \bar{2}^G & 0 \\ 0 & -\bar{3}^L \cdot \bar{3}^G & 0 \\ -\bar{2}^C \cdot \bar{1}^G & 0 & -\bar{3}^C \cdot \bar{1}^G \\ -\bar{2}^C \cdot \bar{2}^G & 0 & -\bar{3}^C \cdot \bar{2}^G \\ -\bar{2}^C \cdot \bar{3}^G & 0 & -\bar{3}^C \cdot \bar{3}^G \end{bmatrix} \begin{bmatrix} \rho(\dot{\lambda})^2 \\ \dot{\lambda} \\ a_V \end{bmatrix} \qquad (4)$$

Examination of Constraints

This set of six equations has a solution if and only if the determinant of the 6X6 matrix of coefficients of the six dependent variables is not zero. For example, suppose we choose as the independent variables the three coordinates of angular velocity, ω_{1G}, ω_{2G}, and ω_{3G}, and one coordinate of linear acceleration, a_iG. Then the dependent variables will be $\dot{\gamma}_1$, $\dot{\gamma}_2$, $\dot{\gamma}_3$, $\rho\ddot{\lambda}$, a_jG, and a_kG (i, j, k = 1, 2, 3 but $i \neq j \neq k \neq i$). The matrix of coefficients of the dependent variables is

$$\begin{bmatrix} 1 & 0 & \bar{3}^C \cdot \bar{1}^G & 0 & 0 & 0 \\ 0 & \bar{2}^M \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G & 0 & 0 & 0 \\ 0 & \bar{2}^M \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{1}^G & -\delta_{1_j} & -\delta_{1_k} \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{2}^G & -\delta_{2_j} & -\delta_{2_k} \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{3}^G & -\delta_{3_j} & -\delta_{3_k} \end{bmatrix} \qquad (5)$$

where

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

The determinant of (5) is

$$\left[(\bar{2}^M \cdot \bar{2}^G)(\bar{3}^C \cdot \bar{3}^G) - (\bar{2}^M \cdot \bar{3}^G)(\bar{3}^C \cdot \bar{2}^G)\right] \Big[\bar{1}^C \cdot \bar{1}^G(\delta_{2_j}\delta_{3_k} - \delta_{3_j}\delta_{2_k}) - \bar{1}^C \cdot \bar{2}^G(\delta_{1_j}\delta_{3_k} - \delta_{3_j}\delta_{1_k}) + \bar{1}^C \cdot \bar{3}^G(\delta_{1_j}\delta_{2_k} - \delta_{2_j}\delta_{1_k})\Big]$$

So for the equations to have a solution (i.e., for the simulation to be realizable), the following conditions must prevail:

$$(\bar{2}^M \cdot \bar{2}^G)(\bar{3}^C \cdot \bar{3}^G) - (\bar{2}^M \cdot \bar{3}^G)(\bar{3}^C \cdot \bar{2}^G) \neq 0$$

and

$$\bar{1}^C \cdot \bar{1}^G(\delta_{2_j}\delta_{3_k} - \delta_{3_j}\delta_{2_k}) - \bar{1}^C \cdot \bar{2}^G(\delta_{1_j}\delta_{3_k} - \delta_{3_j}\delta_{1_k}) + \bar{1}^C \cdot \bar{3}^G(\delta_{1_j}\delta_{2_k} - \delta_{2_j}\delta_{1_k}) \neq 0$$

To interpret the first condition it is necessary to convert the dot products to trigonometric functions. The transformation matrices of equations (1) and (2) represent successive rotations of coordinates through the angles λ, γ_3, γ_2, and γ_1, in that order. Then (according to ref. 1, pp. 6-10),

$$\begin{bmatrix} \bar{1}^{L}\cdot\bar{1}^{C} & \bar{2}^{L}\cdot\bar{1}^{C} & \bar{3}^{L}\cdot\bar{1}^{C} \\ \bar{1}^{L}\cdot\bar{2}^{C} & \bar{2}^{L}\cdot\bar{2}^{C} & \bar{3}^{L}\cdot\bar{2}^{C} \\ \bar{1}^{L}\cdot\bar{3}^{C} & \bar{2}^{L}\cdot\bar{3}^{C} & \bar{3}^{L}\cdot\bar{3}^{C} \end{bmatrix} = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{1}^{C}\cdot\bar{1}^{M} & \bar{2}^{C}\cdot\bar{1}^{M} & \bar{3}^{C}\cdot\bar{1}^{M} \\ \bar{1}^{C}\cdot\bar{2}^{M} & \bar{2}^{C}\cdot\bar{2}^{M} & \bar{3}^{C}\cdot\bar{2}^{M} \\ \bar{1}^{C}\cdot\bar{3}^{M} & \bar{2}^{C}\cdot\bar{3}^{M} & \bar{3}^{C}\cdot\bar{3}^{M} \end{bmatrix} = \begin{bmatrix} \cos\gamma_3 & \sin\gamma_3 & 0 \\ -\sin\gamma_3 & \cos\gamma_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{1}^{M}\cdot\bar{1}^{J} & \bar{2}^{M}\cdot\bar{1}^{J} & \bar{3}^{M}\cdot\bar{1}^{J} \\ \bar{1}^{M}\cdot\bar{2}^{J} & \bar{2}^{M}\cdot\bar{2}^{J} & \bar{3}^{M}\cdot\bar{2}^{J} \\ \bar{1}^{M}\cdot\bar{3}^{J} & \bar{2}^{M}\cdot\bar{3}^{J} & \bar{3}^{M}\cdot\bar{3}^{J} \end{bmatrix} = \begin{bmatrix} \cos\gamma_2 & 0 & -\sin\gamma_2 \\ 0 & 1 & 0 \\ \sin\gamma_2 & 0 & \cos\gamma_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{1}^{J}\cdot\bar{1}^{G} & \bar{2}^{J}\cdot\bar{1}^{G} & \bar{3}^{J}\cdot\bar{1}^{G} \\ \bar{1}^{J}\cdot\bar{2}^{G} & \bar{2}^{J}\cdot\bar{2}^{G} & \bar{3}^{J}\cdot\bar{2}^{G} \\ \bar{1}^{J}\cdot\bar{3}^{G} & \bar{2}^{J}\cdot\bar{3}^{G} & \bar{3}^{J}\cdot\bar{3}^{G} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma_1 & \sin\gamma_1 \\ 0 & -\sin\gamma_1 & \cos\gamma_1 \end{bmatrix}$$

Furthermore, any number of successive rotations can be combined. For example,

$$\begin{bmatrix} \bar{1}^{C}\cdot\bar{1}^{G} & \bar{2}^{C}\cdot\bar{1}^{G} & \bar{3}^{C}\cdot\bar{1}^{G} \\ \bar{1}^{C}\cdot\bar{2}^{G} & \bar{2}^{C}\cdot\bar{2}^{G} & \bar{3}^{C}\cdot\bar{2}^{G} \\ \bar{1}^{C}\cdot\bar{3}^{G} & \bar{2}^{C}\cdot\bar{3}^{G} & \bar{3}^{C}\cdot\bar{3}^{G} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma_1 & \sin\gamma_1 \\ 0 & -\sin\gamma_1 & \cos\gamma_1 \end{bmatrix} \begin{bmatrix} \cos\gamma_2 & 0 & -\sin\gamma_2 \\ 0 & 1 & 0 \\ \sin\gamma_2 & 0 & \cos\gamma_2 \end{bmatrix} \begin{bmatrix} \cos\gamma_3 & \sin\gamma_3 & 0 \\ -\sin\gamma_3 & \cos\gamma_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of equation (2) now can be written

$$\begin{bmatrix} \cos\gamma_2\cos\gamma_3 & \cos\gamma_2\sin\gamma_3 & -\sin\gamma_2 \\ \sin\gamma_1\sin\gamma_2\cos\gamma_3 - \cos\gamma_1\sin\gamma_3 & \sin\gamma_1\sin\gamma_2\sin\gamma_3 + \cos\gamma_1\cos\gamma_3 & \sin\gamma_1\cos\gamma_2 \\ \cos\gamma_1\sin\gamma_2\cos\gamma_3 + \sin\gamma_1\sin\gamma_3 & \cos\gamma_1\sin\gamma_2\sin\gamma_3 - \sin\gamma_1\cos\gamma_3 & \cos\gamma_1\cos\gamma_2 \end{bmatrix}$$

and that of equation (3)

$$\begin{bmatrix} 1 & 0 & -\sin\gamma_2 & -\sin\gamma_2 \\ 0 & \cos\gamma_1 & \sin\gamma_1\cos\gamma_2 & \sin\gamma_1\cos\gamma_2 \\ 0 & -\sin\gamma_1 & \cos\gamma_1\cos\gamma_2 & \cos\gamma_1\cos\gamma_2 \end{bmatrix}$$

Then (5), which is composed of elements of the matrices of equations (2) and (3), becomes

$$\begin{bmatrix} 1 & 0 & -\sin\gamma_2 & 0 & 0 & 0 \\ 0 & \cos\gamma_1 & \sin\gamma_1\cos\gamma_2 & 0 & 0 & 0 \\ 0 & -\sin\gamma_1 & \cos\gamma_1\cos\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma_2\cos\gamma_3 & -\delta_{1_j} & -\delta_{1_k} \\ 0 & 0 & 0 & \sin\gamma_1\sin\gamma_2\cos\gamma_3 - \cos\gamma_1\sin\gamma_3 & -\delta_{2_j} & -\delta_{2_k} \\ 0 & 0 & 0 & \cos\gamma_1\sin\gamma_2\cos\gamma_3 + \sin\gamma_1\sin\gamma_3 & -\delta_{3_j} & -\delta_{3_k} \end{bmatrix}$$

Now the first condition necessary for the determinant of (5) to be other than zero becomes

$$(\cos\gamma_1)(\cos\gamma_1\cos\gamma_2) - (-\sin\gamma_1)(\sin\gamma_1\cos\gamma_2) = \cos\gamma_2 \neq 0$$

Therefore

$$\gamma_2 \neq \pm\pi/2$$

This condition says that the cockpit drive axis, $\bar{1}^J$, must not be alined with the outer gimbal drive axis $\bar{3}^C$.

The second condition is readily interpretable in dot-product form. It depends on which linear acceleration component, in addition to ω_{1G}, ω_{2G}, and ω_{3G}, is independent. If a_{1G} is independent $\bar{1}^C \cdot \bar{1}^G \neq 0$; that is, the $\bar{1}^G$ axis cannot lie in the plane perpendicular to the $\bar{1}^C$ axis. If a_{2G} is independent, $\bar{2}^G$ cannot be perpendicular to $\bar{1}^C$. If a_{3G} is independent, $\bar{3}^G$ cannot be perpendicular to $\bar{1}^C$. In general, if $\bar{1}^C \cdot \bar{i}^G = 0$, then the independence of a_{iG} no longer exists and, like a_{jG} and a_{kG}, it is dependent upon the coordinates of $\bar{\Omega}$; that is, all components of $\bar{A}$ are dependent on $\bar{\Omega}$.

Equation (4) can be written in a more general form applicable to the case ω_{1G}, ω_{2G}, ω_{3G}, and a_{1G} independent.

$$
\begin{bmatrix}
1 & 0 & -\sin\gamma_2 & 0 & 0 & 0 \\
0 & \cos\gamma_1 & \sin\gamma_1\cos\gamma_2 & 0 & 0 & 0 \\
0 & -\sin\gamma_1 & \cos\gamma_1\cos\gamma_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\gamma_2\cos\gamma_3 & -\delta_{1_j} & -\delta_{1_k} \\
0 & 0 & 0 & \sin\gamma_1\sin\gamma_2\cos\gamma_3 - \cos\gamma_1\sin\gamma_3 & -\delta_{2_j} & -\delta_{2_k} \\
0 & 0 & 0 & \cos\gamma_1\sin\gamma_2\cos\gamma_3 + \sin\gamma_1\sin\gamma_3 & -\delta_{3_j} & \delta_{3_k}
\end{bmatrix}
\begin{bmatrix}
\dot\gamma_1 \\ \dot\gamma_2 \\ \dot\gamma_3 \\ \rho\ddot\lambda \\ a_{jG} \\ a_{kG}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \sin\gamma_2 & 0 \\
0 & 1 & 0 & 0 & 0 & -\sin\gamma_1\cos\gamma_2 & 0 \\
0 & 0 & 1 & 0 & 0 & -\cos\gamma_1\cos\gamma_2 & 0 \\
0 & 0 & 0 & \delta_{1_i} & -\cos\gamma_2\sin\gamma_3 & 0 & \sin\gamma_2 \\
0 & 0 & 0 & \delta_{2_i} & -\sin\gamma_1\sin\gamma_2\sin\gamma_3 - \cos\gamma_1\cos\gamma_3 & 0 & -\sin\gamma_1\cos\gamma_2 \\
0 & 0 & 0 & \delta_{3_i} & -\cos\gamma_1\sin\gamma_2\sin\gamma_3 + \sin\gamma_1\cos\gamma_3 & 0 & -\cos\gamma_1\cos\gamma_2
\end{bmatrix}
\begin{bmatrix}
\omega_{1G} \\ \omega_{2G} \\ \omega_{3G} \\ a_{iG} \\ \rho(\dot\lambda)^2 \\ \dot\lambda \\ a_v
\end{bmatrix}
\tag{6}
$$

where $i, j, k = 1, 2, 3$ but $i \neq j \neq k \neq i$;

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_2 \neq \pi/2$$

$$\bar{1}^G \cdot \bar{1}^C \neq 0$$

EQUATIONS FOR REPRODUCING PART OF THE ANGULAR VELOCITY AND ALL OR PART OF THE LINEAR ACCELERATION

Development of the Equations

Unfortunately, only for the case in which the independent variables are made up of all three of the coordinates of angular velocity and one of the coordinates of linear acceleration do equations (6) have a solution. Otherwise the determinant of the matrix of coefficients of the dependent variables is zero. We can get around this by using the rate of change of linear acceleration instead of the acceleration itself. In C axes coordinates

$$\dot{\bar{A}} = \rho\dddot\lambda\bar{1}^C + 2\rho\dot\lambda\ddot\lambda\bar{2}^C + \dot a_v\bar{3}^C + (\dot\lambda\bar{3}^C) \times (\rho\ddot\lambda\bar{1}^C + \rho\dot\lambda^2\bar{2}^C + a_v\bar{3}^C)$$

$$= (\rho\dddot\lambda - \rho\dot\lambda^3)\bar{1}^C + (3\rho\dot\lambda\ddot\lambda)\bar{2}^C + \dot a_v\bar{3}^C$$

In G axes coordinates

$$\bar{A} = \dot{a}_{1G}\bar{1}^G + \dot{a}_{2G}\bar{2}^G + \dot{a}_{3G}\bar{3}^G + (\omega_{1G}\bar{1}^G + \omega_{2G}\bar{2}^G + \omega_{3G}\bar{3}^G)$$

$$\times\ (a_{1G}\bar{1}^G + a_{2G}\bar{2}^G + a_{3G}\bar{3}^G)$$

$$= (\dot{a}_{1G} + a_{3G}\omega_{2G} - a_{2G}\omega_{3G})\bar{1}^G + (\dot{a}_{2G} + a_{1G}\omega_{3G} - a_{3G}\omega_{1G})\bar{2}^G$$

$$+ (\dot{a}_{3G} + a_{2G}\omega_{1G} - a_{1G}\omega_{2G})\bar{3}^G$$

where $\times$ indicates the vector product.

The equations for transforming from C axes coordinates to cockpit-axes coordinates have the matrix of equations (2):

$$\begin{bmatrix} \dot{a}_{1G} + a_{3G}\omega_{2G} - a_{2G}\omega_{3G} \\ \dot{a}_{2G} + a_{1G}\omega_{3G} - a_{3G}\omega_{1G} \\ \dot{a}_{3G} + a_{2G}\omega_{1G} - a_{1G}\omega_{2G} \end{bmatrix} = \begin{bmatrix} \bar{1}^C \cdot \bar{1}^G & \bar{2}^C \cdot \bar{1}^G & \bar{3}^C \cdot \bar{1}^G \\ \bar{1}^C \cdot \bar{2}^G & \bar{2}^C \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G \\ \bar{1}^C \cdot \bar{3}^G & \bar{2}^C \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G \end{bmatrix} \begin{bmatrix} \rho\dddot{\lambda} - \rho(\dot{\lambda})^3 \\ 3\rho\dot{\lambda}\ddot{\lambda} \\ \dot{a}_V \end{bmatrix} \qquad (7)$$

We can combine equations (3) and (7) as we did (2) and (3):

$$\begin{bmatrix} 1 & 0 & \bar{3}^C \cdot \bar{1}^G & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \bar{2}^M \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & \bar{2}^M \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{1}^G & -1 & 0 & 0 & 0 & -a_{3G} & a_{2G} \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{2}^G & 0 & -1 & 0 & a_{3G} & 0 & -a_{1G} \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{3}^G & 0 & 0 & -1 & -a_{2G} & a_{1G} & 0 \end{bmatrix} \begin{bmatrix} \dot{\gamma} \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \rho\ddot{\lambda} \\ \dot{a}_{1G} \\ \dot{a}_{2G} \\ \dot{a}_{3G} \\ \omega_{1G} \\ \omega_{2G} \\ \omega_{3G} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\bar{3}^L \cdot \bar{1}^G & 0 \\ 0 & 0 & -\bar{3}^L \cdot \bar{2}^G & 0 \\ 0 & 0 & -\bar{3}^L \cdot \bar{3}^G & 0 \\ \bar{1}^C \cdot \bar{1}^G & -\bar{2}^C \cdot \bar{1}^G & 0 & -\bar{3}^C \cdot \bar{1}^G \\ \bar{1}^C \cdot \bar{2}^G & -\bar{2}^C \cdot \bar{2}^G & 0 & -\bar{3}^C \cdot \bar{2}^G \\ \bar{1}^C \cdot \bar{3}^G & -\bar{2}^C \cdot \bar{3}^G & 0 & -\bar{3}^C \cdot \bar{3}^G \end{bmatrix} \begin{bmatrix} \rho(\dot{\lambda})^3 \\ 3\rho\ddot{\lambda}\dot{\lambda} \\ \dot{\lambda} \\ \dot{a}_V \end{bmatrix} \qquad (8)$$

Examination of Constraints

Again the equations have a solution only if the determinant of the matrix of coefficients of the dependent variables is not zero. The coefficient matrix will be of the form

$$\begin{bmatrix} 1 & 0 & \bar{3}^C \cdot \bar{1}^G & 0 & t_{15} & t_{16} \\ 0 & -\bar{2}^M \cdot \bar{2}^G & \bar{3}^C \cdot \bar{2}^G & 0 & t_{25} & t_{26} \\ 0 & \bar{2}^M \cdot \bar{3}^G & \bar{3}^C \cdot \bar{3}^G & 0 & t_{35} & t_{36} \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{1}^G & t_{45} & t_{46} \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{2}^G & t_{55} & t_{56} \\ 0 & 0 & 0 & \bar{1}^C \cdot \bar{3}^G & t_{65} & t_{66} \end{bmatrix} \qquad (9)$$

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where the t_{ij} are the coefficients of the two dependent G axes coordinates. These two may be either one coordinate of angular velocity and one coordinate of linear acceleration, or two of the coordinates of angular velocity.

The determinant of (9) is of the same form as the determinant of (5), but with the δ_{mn} replaced with the t_{ij}. The determinant of (9) is

$$\left[(\bar{2}^M \cdot \bar{2}^G)(\bar{3}^C \cdot \bar{3}^G) - (\bar{2}^M \cdot \bar{3}^G)(\bar{3}^C \cdot \bar{2}^G)\right] \Big[\bar{1}^C \cdot \bar{1}^G(t_{55}t_{66} - t_{65}t_{56}) - \bar{1}^C \cdot \bar{2}^G(t_{45}t_{66} - t_{66}t_{46}) + \bar{1}^C \cdot \bar{3}^G(t_{45}t_{55} - t_{55}t_{46})\Big]$$

The first condition necessary for the solution of equations (6) is necessary also for the solution of equations (8); that is,

$$\gamma_2 \neq \pm \pi/2$$

The second condition states

$$\bar{1}^C \cdot \bar{1}^G(t_{55}t_{66} - t_{65}t_{56}) - \bar{1}^C \cdot \bar{2}^G(t_{45}t_{66} - t_{66}t_{46}) + \bar{1}^C \cdot \bar{3}^G(t_{45}t_{55} - t_{55}t_{46}) \neq 0$$

We must examine the implications of this last condition for each of 12 combinations of 4 (independent) variables chosen from among the 6 G axes coordinates (we exclude the three combinations which contain all three angular coordinates and which have been accounted for in equations (6)). For example, if a_{1G}, a_{2G}, a_{3G}, and ω_{1G} are independent, the second condition for the determinant of (9) not to be zero is satisfied when

$$\bar{1}^C \cdot \bar{1}^G(a_{1G}^2) - \bar{1}^C \cdot \bar{2}^G(-a_{1G}a_{2G}) + \bar{1}^C \cdot \bar{3}^C(a_{1G}a_{3G}) \neq 0$$

Equivalently,

$$a_{1G} \neq 0$$

and

$$\bar{1}^C \cdot \bar{1}^G(a_{1G}) + \bar{1}^C \cdot \bar{2}^G(a_{2G}) + \bar{1}^C \cdot \bar{3}^G(a_{3G}) \neq 0$$

If equations (2) are multiplied by the inverse of the matrix of (2), it
A497 is apparent that the left side of the last inequality is exactly $\rho\ddot{\lambda}$, one of the dependent variables. So we can compile the following list of combinations of independent variables and attendant restrictions.

Combination 1. a_{1G}, a_{2G}, a_{3G}, ω_{1G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{1G} \neq 0$$

$$\rho\ddot{\lambda} \neq 0$$

Combination 2. a_{1G}, a_{2G}, a_{3G}, ω_{2G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{2G} \neq 0$$

$$\rho\ddot{\lambda} \neq 0$$

Combination 3. a_{1G}, a_{2G}, a_{3G}, ω_{3G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{3G} \neq 0$$

$$\rho\ddot{\lambda} \neq 0$$

Combination 4. a_{1G}, a_{2G}, ω_{1G}, ω_{2G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{1G}^2 + a_{2G}^2 \neq 0$$

If $a_{1G} = 0$, then

$$\bar{2}^G \cdot \bar{1}^C \neq 0$$

If $a_{2G} = 0$, then

$$\bar{1}^G \cdot \bar{1}^C \neq 0$$

or, equivalently,

$$\gamma_2 \neq \pm \pi/2$$

$$\gamma_3 \neq \pm \pi/2$$

Combination 5. a_{1G}, a_{2G}, ω_{1G}, ω_{3G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{3G} \neq 0$$

$$\bar{2}^G \cdot \bar{1}^C \neq 0$$

Combination 6. a_{1G}, a_{2G}, ω_{2G}, ω_{3G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{3G} \neq 0$$

$$\bar{1}^G \cdot \bar{1}^C \neq 0$$

or, equivalently,

$$\gamma_2 \neq \pi/2$$

$$\gamma_3 \neq \pm \pi/2$$

Combination 7. a_{1G},,a_{3G}, ω_{1G}, ω_{2G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{2G} \neq 0$$

$$\bar{3}^G \cdot \bar{1}^C \neq 0$$

Combination 8. a_{1G}, a_{3G}, ω_{1G}, ω_{3G} independent:

$$\gamma_2 = \pm \pi/2$$

$$a_{1G}^2 + a_{3G}^2 \neq 0$$

If $a_{1G} = 0$, then

$$\bar{3}^G \cdot \bar{1}^C \neq 0$$

If $a_{3G} = 0$, then

$$\bar{1}^G \cdot \bar{1}^C \neq 0$$

or, equivalently,

$$\gamma_2 \neq \pm \pi/2$$

$$\gamma_3 \neq \pm \pi/2$$

Combination 9. a_{1G}, a_{3G}, ω_{2G}, ω_{3G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{2G} \neq 0$$

$$\bar{1}^G \cdot \bar{1}^C \neq 0$$

or equivalently,

$$\gamma_2 \neq \pm \pi/2$$

$$\gamma_3 \neq \pm \pi/2$$

Combination 10. a_{2G}, a_{3G}, ω_{1G}, ω_{2G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{1G} \neq 0$$

$$\bar{3}^G \cdot \bar{1}^C \neq 0$$

Combination 11. a_{2G}, a_{3G}, ω_{1G}, ω_{3G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{1G} \neq 0$$

$$\bar{2}^G \cdot \bar{1}^C \neq 0$$

Combination 12. a_{2G}, a_{3G}, ω_{2G}, ω_{3G} independent:

$$\gamma_2 \neq \pm \pi/2$$

$$a_{2G}^2 + a_{3G}^2 \neq 0$$

If $a_{2G} = 0$ then

$$\bar{3}^G \cdot \bar{1}^C \neq 0$$

If $a_{3G} = 0$ then

$$\bar{2}^G \cdot \bar{1}^C \neq 0$$

As before, the restrictions $\bar{i}^G \cdot \bar{1}^C \neq 0$, $i = 1, 2, 3$ constrain the $\bar{i}^G$ axis to be outside the plane perpendicular to the $\bar{1}^C$ axis.

Equations (8) can be rewritten in a form more convenient for solution.

$$\begin{bmatrix} 1 & 0 & -\sin\gamma_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \cos\gamma_1 & \sin\gamma_1\cos\gamma_2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\sin\gamma_1 & \cos\gamma_1\cos\gamma_2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \cos\gamma_2\cos\gamma_3 & -1 & 0 & 0 & 0 & -a_{3G} & a_{2G} \\ 0 & 0 & 0 & \begin{pmatrix}\sin\gamma_1\sin\gamma_2\cos\gamma_3 \\ -\cos\gamma_1\sin\gamma_3\end{pmatrix} & 0 & -1 & 0 & a_{3G} & 0 & -a_{1G} \\ 0 & 0 & 0 & \begin{pmatrix}\cos\gamma_1\sin\gamma_2\cos\gamma_3 \\ +\sin\gamma_1\sin\gamma_3\end{pmatrix} & 0 & 0 & -1 & -a_{2G} & a_{1G} & 0 \end{bmatrix} \begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \rho\dddot{\lambda} \\ \dot{a}_{1G} \\ \dot{a}_{2G} \\ \dot{a}_{3G} \\ \omega_{1G} \\ \omega_{2G} \\ \omega_{3G} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \sin\gamma_2 & 0 \\ 0 & 0 & -\sin\gamma_1\cos\gamma_2 & 0 \\ 0 & 0 & -\cos\gamma_1\cos\gamma_2 & 0 \\ -\cos\gamma_2\cos\gamma_3 & -\cos\gamma_2\sin\gamma_3 & 0 & \sin\gamma_2 \\ \begin{pmatrix}-\sin\gamma_1\sin\gamma_2\cos\gamma_3 \\ +\cos\gamma_1\sin\gamma_3\end{pmatrix} & \begin{pmatrix}-\sin\gamma_1\sin\gamma_2\sin\gamma_3 \\ -\cos\gamma_1\cos\gamma_3\end{pmatrix} & 0 & -\sin\gamma_1\cos\gamma_2 \\ \begin{pmatrix}-\cos\gamma_1\sin\gamma_2\cos\gamma_3 \\ -\sin\gamma_1\sin\gamma_3\end{pmatrix} & \begin{pmatrix}-\cos\gamma_1\sin\gamma_2\sin\gamma_3 \\ +\sin\gamma_1\cos\gamma_3\end{pmatrix} & 0 & -\cos\gamma_1\cos\gamma_2 \end{bmatrix} \begin{bmatrix} \rho(\dot{\lambda})^3 \\ 3\rho\dot{\lambda}\ddot{\lambda} \\ \dot{\lambda} \\ \dot{a}_v \end{bmatrix} \qquad (10)$$

THE INITIAL CONDITIONS

Equations (10) and (6) can be programmed on an analog computer and the output signals representing $\dot{\gamma}_1$, $\dot{\gamma}_2$, $\dot{\gamma}_3$, and $\dot{\lambda}$ used to drive the simulator. Some additional information is needed, however. Initial conditions must be set on these output signals. Initial values of γ_1, γ_2, γ_3, and $\dot{\lambda}$ can be calculated as follows. Assume that any simulation will be started from a condition of constant linear acceleration and constant angular velocity, and that the linear acceleration will be specified. Then

$$\bar{A}_o = a_{1G_o}\bar{1}^G + a_{2G_o}\bar{2}^G + a_{3G_o}\bar{3}^G = \rho(\dot{\lambda}_o)^2\bar{2}^C - g\bar{3}^C \qquad (11)$$

$$\overline{\Omega}_o = \omega_{1G_o}\overline{1}^G + \omega_{2G_o}\overline{2}^G + \omega_{3G_o}\overline{3}^G = \dot{\lambda}_o\overline{3}^C \qquad (12)$$

Where the subscript o denotes an initial condition, and g is the acceleration due to gravity. According to the assumption then,

$$\dot{\gamma}_{1_o} = \dot{\gamma}_{2_o} = \dot{\gamma}_{3_o} = \ddot{\lambda} = \dddot{\lambda} = 0$$

and

$$\rho(\dot{\lambda}_o)^2 = \sqrt{a_{1G_o}{}^2 + a_{2G_o}{}^2 + a_{3G_o}{}^2 - g^2}$$

We must now find values of γ_{1o}, γ_{2o}, and γ_{3o} which satisfy equations (11) and (12). Because of the assumptions of steady state of $\overline{A}_o$ and $\overline{\Omega}_o$, we have one redundancy among the three angles γ_{1o}, γ_{2o}, and γ_{3o}. In other words, we can arbitrarily choose an initial value for one of them, and then solve for the other two. In the case of the Ames simulator the angle most advantageously chosen arbitrarily is γ_{2_o}, since γ_2 is restricted to about ±50° displacement. To solve for γ_{1o} and γ_{3o} in terms of γ_2 we can use equations (2). Setting $\rho\ddot{\lambda} = 0$ and $a_V = -g$, and writing the dot products as their trigonometric equivalents, we have the three equations:

$$\left.\begin{aligned}
a_{1G_o} &= -\rho(\dot{\lambda}_o)^2\cos\gamma_{2_o}\sin\gamma_{3_o} - g\sin\gamma_{2_o} \\
a_{2G_o} &= -\rho(\dot{\lambda}_o)^2(\sin\gamma_{1_o}\sin\gamma_{2_o}\sin\gamma_{3_o} + \cos\gamma_{1_o}\cos\gamma_{3_o}) \\
&\qquad + g\sin\gamma_{1_o}\cos\gamma_{2_o} \\
a_{3G_o} &= -\rho(\dot{\lambda}_o)^2(\cos\gamma_{1_o}\sin\gamma_{2_o}\sin\gamma_{3_o} - \sin\gamma_{1_o}\cos\gamma_{3_o}) \\
&\qquad + g\cos\gamma_{1_o}\cos\gamma_{2_o}
\end{aligned}\right\} \qquad (13)$$

The first of these equations can be solved for γ_{3o}, since every other variable in it is known. Then each of these two values can be used to solve simultaneously the second and third equations for γ_{1o}. Multiplying the second of equations (13) by $(-\cos\gamma_{1_o})$ and the third by $(\sin\gamma_{1o})$, adding, and transposing yields:

$$a_{3G_o}\sin\gamma_{1_o} = \rho(\dot{\lambda}_o)^2\cos\gamma_{3_o} + a_{2G_o}\cos\gamma_{1_o} \qquad (14)$$

Then, if equation (14) is squared and $(1 - \cos^2\gamma_{1_o})$ substituted for $\sin^2\gamma_{1o}$,

$$(a_{2G_o}{}^2 + a_{3G_o}{}^2)\cos^2\gamma_{1_o} + 2a_{2G_o}\rho(\dot{\lambda}_o)^2\cos\gamma_{3_o}\cos\gamma_{1_o} + [\rho(\dot{\lambda}_o)^2]^2\cos^2\gamma_{3_o} - a_{3G_o}{}^2 = 0$$

Then

$$\cos \gamma_{1_o} = \frac{-a_{2G_o}\rho(\dot{\lambda}_o)^2\cos \gamma_{3_o} \pm a_{3G_o}\sqrt{a_{2G_o}{}^2 + a_{3G_o}{}^2 - [\rho(\dot{\lambda}_o)^2]^2\cos^2\gamma_{3_o}}}{a_{2G_o}{}^2 + a_{3G_o}{}^2} \tag{15}$$

For each of the values of γ_{3_o} obtained from the first of equations (13), two values of γ_{1_o} will be obtained from (15), only one of which, the correct one, satisfies (14).

AXES OF THE SIMULATED VEHICLE

Throughout the development of the equations the simulator cockpit axes were named $\bar{1}^G$, $\bar{2}^G$, $\bar{3}^G$ instead of being given some x, y, z relationship to axes of the vehicle to be simulated. This was done for the following reason. Let $\bar{x}^S$, $\bar{y}^S$, and $\bar{z}^S$ be unit vectors associated, respectively, with the longitudinal axis, the lateral axis, and the vertical axis of an axis system fixed in the simulated vehicle. The axes of the simulated vehicle can be related to the axes of the simulator cockpit in three ways:

Orientation X_1,

$$\bar{x}^S = \bar{1}^G$$

$$\bar{y}^S = \bar{2}^G$$

$$\bar{z}^S = \bar{3}^G$$

Orientation X_2,

$$\bar{x}^S = \bar{2}^G$$

$$\bar{y}^S = \bar{3}^G$$

$$\bar{z}^S = \bar{1}^G$$

Orientation X_3,

$$\bar{x}^S = \bar{3}^G$$

$$\bar{y}^S = \bar{1}^G$$

$$\bar{z}^S = \bar{2}^G$$

Of course, the pilot and the interior of the simulator cockpit must be oriented according to the S axes system. Depending on the problem to be simulated, that is, the independent variables, one orientation may be more desirable than the others, or there may be no difference among them. For example, suppose for a particular problem it is necessary to faithfully simulate a_y, a_z, p, and q. Then, if X_1 were chosen a_{2G}, a_{3G}, ω_{1G}, and ω_{2G} would be the independent variables. This is Combination 10 of the preceding section. The restrictions on the simulation then are:

$$\gamma_2 \neq \pm \pi/2$$

$$ax \neq 0$$

$$\bar{z}^S \cdot \bar{1}^C \neq 0$$

If X_2 were chosen, a_{3G}, a_{1G}, ω_{2G}, and ω_{3G} would be independent. This is Combination 9, with restrictions

$$\gamma_2 \neq \pm \pi/2$$

$$ax \neq 0$$

$$\gamma_3 \neq \pm \pi/2$$

If X_3 were chosen, a_{1G}, a_{2G}, ω_{3G} and ω_{1G} would be independent. This is Combination 5, with restrictions

$$\gamma_2 \neq \pm \pi/2$$

$$ax \neq 0$$

$$\bar{z}^S \cdot \bar{1}^C \neq 0$$

Which of the three arrangements is best will depend on the particular problem and on the physical limitations of the simulator.

CONCLUDING REMARKS

The coordinate-transformation equations developed in this report apply in particular to the NASA-Ames five-degree-of-freedom piloted flight simulator. However, the method for developing the equations and examining the singularities can be applied to other similar simulators with differently ordered gimbals or a different number of degrees of freedom or both.

For reasons explained in the text, the equations were developed for four degrees of freedom instead of five. It appears possible, within the restrictions imposed by singularities in the equations, to reproduce on the simulator any combination of four of the six components of angular and linear motion of an actual vehicle. Examination of the coefficients in the equations indicates those positions of the gimbals which correspond to singular points of the solutions; so, in practice the simulation can be set up to avoid those positions.

It appears also that the effects of the singularities may be avoided, in some cases, by proper orientation of the pilot with respect to the gimbals.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Oct. 11, 1961

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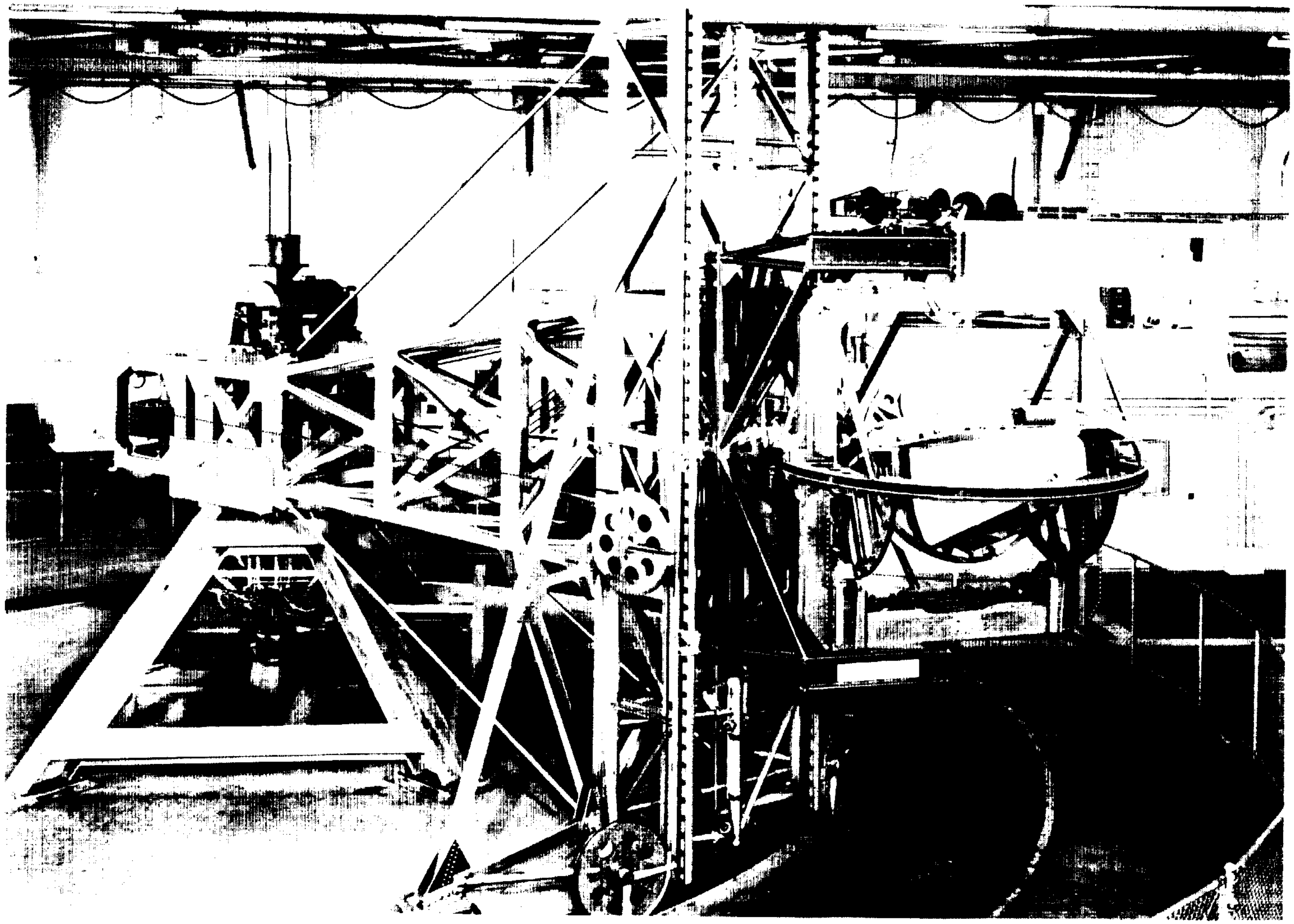

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Figure 1.- The Ames five-degree-of-freedom piloted flight simulator.

A
4
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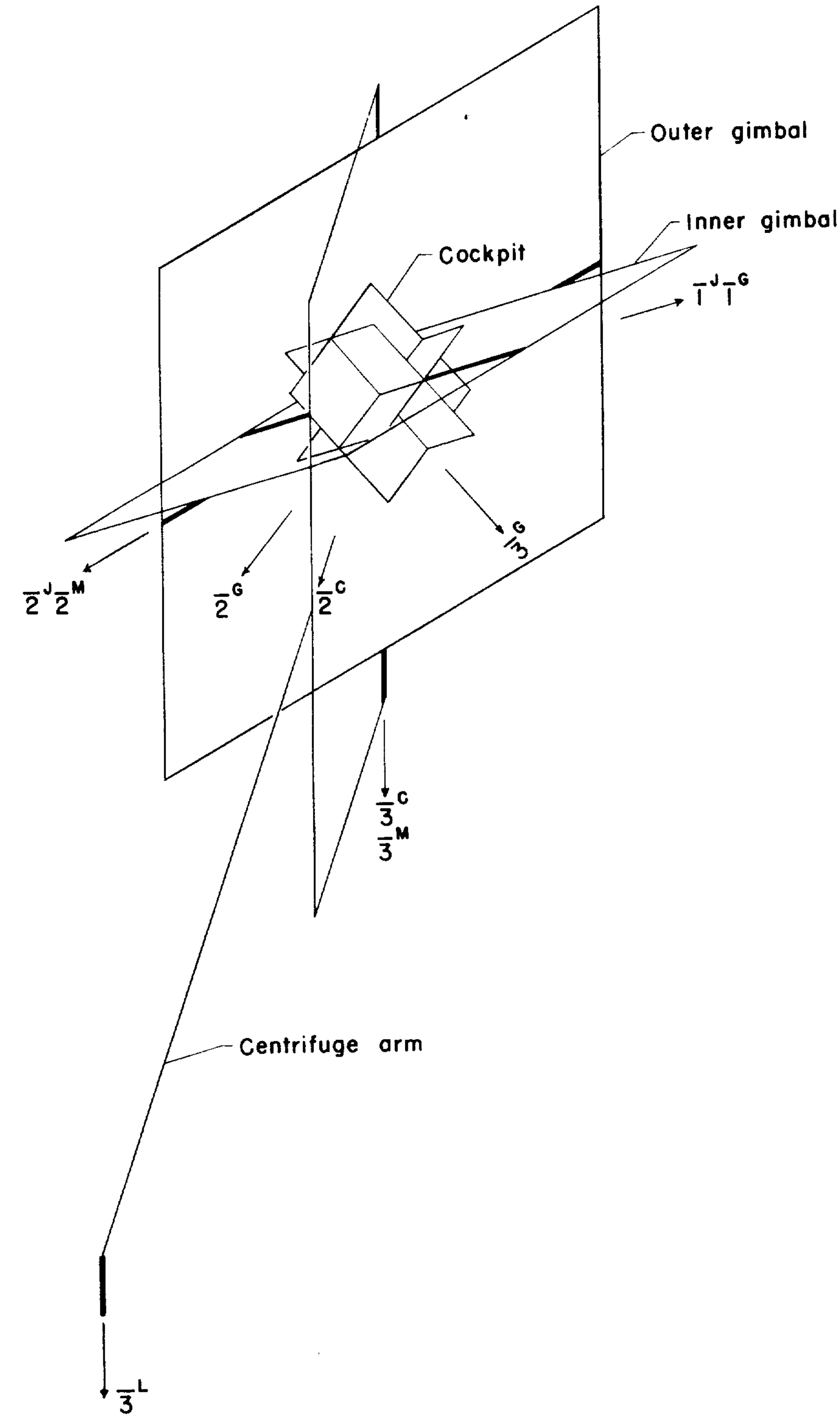

Figure 2.- The coordinate systems of the simulator.